Q.1 If $(a-ib)(x+iy) = (a^2+b^2)i$ and $a+ib \neq 0$, show that $x = b$ and $y = a$.

Sol. Let $Z = a+ib$, then $Z\bar{Z} = a^2+b^2$

Now, $(a+ib)(x-iy) = (a^2+b^2)i$

$\Rightarrow Z(x+iy) = Z\bar{Z}\,i$

$\Rightarrow x+iy = \bar{Z}\,i = (a-ib)i = ai+b$ [by definition of equality of Compex Numbers]

$\Rightarrow x = b, y = a$

Q.2 If $(\text{Cos}+i\,\text{Sin}\theta)^2 = x+iy$, that show $x^2+y^2 = 1$.

Sol. $\left|(\cos\theta+i\sin\theta)^2\right| = |x+iy|$

$|\cos\theta+i\sin\theta|^2 = |x+iy|$

$\Rightarrow |\cos\theta+i\sin\theta| = \sqrt{x^2+y^2}$

$\Rightarrow \left(\sqrt{\cos^2\theta+\sin^2\theta}\right) = \sqrt{x^2+y^2}$

$\Rightarrow x^2+y^2 = 1$

Q.3 If $z_1 = 1+i$, $z_2 = 2-i$, find $\left|\dfrac{z_1-z_2-i}{z_1+z_2+1}\right|$.

Ans. Here $z_1 = 1+i$ & $z_2 = 2-i$

$$\therefore \left|\frac{z_1-z_2-i}{z_1+z_2+1}\right| = \left|\frac{1+i-2+i-i}{1+i+2-i+1}\right|$$

$$= \left|\frac{i-1}{4}\right|$$

$$= \sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}$$

$$= \sqrt{\frac{1}{16}+\frac{1}{16}}$$

$$= \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}}.$$

Q.4 If ω is a complex cube root of unity, prove that

(i) $1+\omega+\omega^2=0$

(ii) $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

(June-2005)

Ans. ω is a complex cube root of unity.

$$\omega=\frac{-1+i\sqrt{3}}{2}$$

(i) $\omega^3=1$

$\therefore \omega^3-1=0$

$\therefore (\omega-1)(\omega^2+\omega+1)=0$

$\therefore \omega=1$ or $\omega^2+\omega+1=0$

$$\therefore \omega=1 \quad or \quad \omega=\frac{-1\pm\sqrt{-3}}{2}$$

$$=\frac{-1\pm\sqrt{3}i}{2}$$

$$If\ \omega=\frac{-1+\sqrt{3}i}{2}$$

$$\omega^2=\frac{-1-\sqrt{3}i}{2}$$

$$\therefore 1+\omega+\omega^2=1+\left(\frac{-1+\sqrt{-3}}{2}\right)+\left(\frac{-1-\sqrt{-3}}{2}\right)=1-\frac{1}{2}-\frac{1}{2}=0$$

(ii) $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}$

$$=\frac{(1+\omega)(2+\omega)+(1+2\omega)(1+\omega)-(1+2\omega)(2+\omega)}{(1+\omega)(2+\omega)(1+2\omega)}$$

$$=\frac{2+3\omega+\omega^2+1+3\omega+2\omega^2-(2+5\omega+2\omega^2)}{(1+\omega)(2+\omega)(1+2\omega)}$$

$$=\frac{1+\omega+\omega^2}{(1+\omega)(2+\omega)(1+2\omega)}$$

$=0$, *by (i) above.*

Q.5 Find the complex conjugate of $\frac{3+5i}{1+2i}$.

Ans. $\frac{3+5i}{1+2i}=\frac{3+5i}{1+2i}\times\frac{1-2i}{1-2i}$

$$=\frac{3-i-10i^2}{1+4} \qquad =\frac{13-i}{5} \qquad =\frac{13}{5}-\frac{i}{5}$$

Hence the complex conjugate of given complex number is $\frac{13}{5}+\frac{i}{5}$

Q.6 Find all $z \in C$ such that $z^4 = -8(1+i\sqrt{3})$

Ans. $z^4 = -8(1+i\sqrt{3})$

$= -8-8\sqrt{3}i$

Here, $a = -8$, $b = -8\sqrt{3}$

$\therefore r = \sqrt{a^2+b^2}$

$= \sqrt{64+64\times 3}$

$= \sqrt{4\times 64}$

$= 2\times 8$

$= 16.$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

$\therefore z^4 = -\left(16\left(\cos\left(\tan^{-1}\sqrt{3}\right)+i\sin\left(\tan^{-1}\sqrt{3}\right)\right)\right)$

$= -\left(16\left(\cos\left(\frac{\pi}{3}\right)+i\sin\frac{\pi}{3}\right)\right) = -16e^{i\pi/3}$

$\therefore z^4 = i^2 16 e^{i\pi/3}$

$\therefore z = 2\sqrt{i}\, e^{i\pi/2}$ (exponential form)

Q.7 Prove that $\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$

Ans. $\cos 4\theta = \cos(2\theta+\theta)$

$\cos 3\theta \cdot \cos\theta - \sin 3\theta \cdot \sin\theta$

$= (4\cos^3\theta - 3\cos\theta)\cos\theta - (3\sin\theta - 4\sin^3\theta)\cdot\sin\theta$

$= 4\cos^4\theta - 3\cos^2\theta - 3\sin^2\theta + 4\sin^4\theta$

$= \cos^4\theta + 3\cos^4\theta - 3\cos^2\theta - 3\sin^2\theta + 3\sin^4\theta + \sin^4\theta$

$= \cos^4\theta + 3\cos^2\theta \cdot \sin^2\theta - 3\sin^2\theta \cdot \cos^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta \cdot \sin^2\theta + \sin^4\theta$

Q.8 Prove that $\cos 5\theta = \cos^5\theta\,(1 - 10\tan^2\theta + 5\tan^4\theta)$

Ans. $\cos 5\theta = \cos(3\theta+2\theta)$

$= \cos 3\theta \cdot \cos 2\theta - \sin 3\theta \cdot \sin 2\theta$

$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) - (3\sin\theta - 4\sin^3\theta)2\sin\theta\cos\theta$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\sin^2\theta\cdot\cos\theta + 8\sin^4\theta\cdot\cos\theta$

$$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$= \cos^5\theta + 15\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$= \cos^5\theta + 5\cos\theta\left(3\cos^4\theta - 4\cos^2\theta + 1\right)$$

$$= \cos^5 + 5\cos\theta\left(\frac{3}{\sec^4\theta} - \frac{4}{\sec^2\theta} + 1\right)$$

$$= \cos^5\theta + \frac{5\cos\theta}{\sec^4\theta}\left(3 - 4\sec^2\theta + \sec^4\theta\right)$$

$$= \cos^5\theta + 5\cos^5\theta\left(3 - 4\left(1 + \tan^2\theta\right) + \left(1 + \tan^2\theta\right)^2\right)$$

$$= \cos^5\theta + 5\cos^5\theta\left[\tan^4\theta - 2\tan^2\theta\right]$$

$$= \cos^5\theta + 5\cos^5\theta.\tan^4\theta - 10\cos^5\theta.\tan^2\theta$$

$$= \cos^5\theta\left(1 + 5\tan^4\theta - 10\tan^2\theta\right)$$

Q.9 Find the square root of 'i'.

Ans. We want to find the square root of i.
Let $z^2 = i$.

The polar form of i is $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$.

$$z^2 = r^2 e^{2i\theta}$$

$$= r^2(\cos\theta + i\sin\theta)^2$$

$$= r^2(\cos 2\theta + i\sin 2\theta),$$ *by De Moivre's theorem*

Thus,

$$r^2(\cos 2\theta + i\sin 2\theta) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Comparing r & θ, $r^2 = 1 \Rightarrow r = \pm 1$.

$$2\theta = \frac{\pi}{2} + 2k\pi,\ k \in Z$$

$$\therefore \theta = \frac{\pi}{4} + k\pi,\ k \in Z$$

$$\therefore z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \quad \& \ z = -\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore z = \frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}} \quad \& \ z = -\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)$$

$$\therefore z = \frac{1+i}{\sqrt{2}} \ \& \ z = -\left(\frac{1+i}{\sqrt{2}}\right)$$

$$\therefore \sqrt{i} = \pm\left(\frac{1+i}{\sqrt{2}}\right)$$

Q.10 If z is the product of two complex numbers z_1 and z_2, prove that

$$|z| = |z_1|\,|z_2|$$

and Arg x = Arg z_1 + Arg z_2

Ans. $z = z_1 : z_2$, where $z_1, z_2 \in C$

Let

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

$$|z_1| = r_1, \qquad |z_2| = r_2$$

$$z = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\therefore |z| = r_1 r_2 = |z_1|\,|z_2|$$

$$Also,\ Arg\, z_1 = \theta_1,\ Arg\, z_2 = \theta_2$$

$$Arg\, z = \theta_1 + \theta_2$$

$$= Arg\, z_1 + Arg\, z_2$$

Q.11 For any three complex numbers z_1, z_2, z_3, prove that

$$z_1\, \text{Im}(\bar{z}_2\, z_3) + z_2\, \text{Im}(\bar{z}_3\, z_1) + z_3\, \text{Im}(\bar{z}_1\, z_2) = 0$$

Ans. Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, $z_3 = x_3+iy_3$

$$\bar{z}_2\, z_3 = (x_2-iy_2)(x_3+iy_3) = x_2x_3 + y_2y_3 + i(x_2y_3 - x_3y_2)$$

$$\therefore \text{Im}(\bar{z}_2\, z_3) = x_2y_3 - x_3y_2$$

Similarly, $\text{Im}(\bar{z}_3 z_1) = x_3y_1 - x_1y_3$

$$\text{Im}(\bar{z}_1 z_2) = x_1y_2 - x_2y_1$$

Now, $z_1\, \text{Im}(\bar{z}_2\, z_3) + z_2\, \text{Im}(\bar{z}_3\, z_1) + z_3\, \text{Im}(\bar{z}_1\, z_2)$

$$= (x_1+iy_1)(x_2y_3-x_3y_2) + (x_2+iy_2)(x_3y_1-x_1y_3) + (x_3+iy_3)(x_1y_2-x_2y_1)$$

$$= x_1x_2y_3 - x_1x_3y_2 + x_2x_3y_1 - x_1x_2y_3 + x_1x_3y_2 - x_2x_3y_1 +$$

$$i(x_2y_1y_3 - x_3y_1y_2 + x_3y_1y_2 - x_1y_2y_3 + x_1y_2y_3 - x_2y_1y_3)$$

$$= 0$$

Q.12 Express $\frac{(1+i)(2+i)}{3+i}$ in the form a + ib.

Ans. $\frac{(1+i)(2+i)}{3+i} \times \frac{3-i}{3-i}$

$$= \frac{(2+3i-1)(3-i)}{3^2+i}$$

$$= \frac{(1+3i)(3-i)}{10}$$

$$= \frac{3+8i+3}{10}$$

$$= \frac{6+8i}{10}$$

$$= \frac{3+4i}{5}$$

$$= \frac{3}{5} + \frac{4}{5}i$$

$$\therefore a = \frac{3}{5},\ b = \frac{4}{5}.$$

Q.13 Apply De Moivre's theorem, prove that **(June-09)**

(i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(ii) $\sin 2\theta = 2\sin\theta\cos\theta$.

Ans. De-Moivre theorem:

$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$, where $m \in Z$

(i) Put m = 2

$\therefore (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$\therefore \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$

$\therefore \cos^2\theta - \sin^2\theta + i\sin 2\theta = \cos 2\theta + i\sin 2\theta$

$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta$

(ii) $i\sin 2\theta = (\cos\theta + i\sin\theta)^2 - \cos 2\theta$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta - \cos 2\theta$

$= \cos 2\theta - \cos 2\theta + 2i\sin\theta\cos\theta$

$= i.2\sin\theta\cos\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

Q.14 Express $\frac{3+4i}{2-3i}$ in the form $r(\cos\theta + i\sin\theta)$. **(Dec-09)**

Ans. $\frac{3+4i}{2-3i} = \frac{3+4i}{2-3i} \times \frac{2+3i}{2+3i}$

$$= \frac{6+17i-12}{4+9}$$

$$= \frac{17i-6}{13}$$

$$= \frac{-6}{13} + i.\frac{17}{13}.$$

Comparing with a+*ib*, $a = \frac{-6}{13}, b = \frac{17}{13}$

$$r = \sqrt{a^2 + b^2} = \sqrt{\frac{36}{169} + \frac{289}{169}}$$

$$= \sqrt{\frac{325}{169}} = \frac{5\sqrt{13}}{13} = \frac{5}{\sqrt{13}}$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\left(\frac{-17}{6}\right)$$

$$\therefore \text{Required form} = \frac{5}{\sqrt{13}}\left[\cos\left\{\tan^{-1}\left(\frac{-17}{6}\right)\right\} + i\sin\left\{\tan^{-1}\left(\frac{-17}{6}\right)\right\}\right]$$

Q.15 Expand $(\cos^6\theta - \sin^6\theta)$ in terms of cosines of multiples of θ.

Ans. $\cos^6\theta - \sin^6\theta$

Let $Z = \cos\theta + i\sin\theta$

$\therefore Z^m = (\cos\theta + i\sin\theta)^m$, where m is a non-zero integer.

$= \cos m\theta + i\sin m\theta$ by De-Moivre's theorem.

Further, $Z^m + Z^{-m}$

$= (\cos m\theta + i\sin m\theta) + (\cos m\theta - i\sin m\theta)$

$= 2\cos m\theta.$

$(2\cos\theta)^6 = (Z+Z^{-1})^6$

$$= Z^6 + \frac{1}{Z^6} + 6\left(Z^4 + \frac{1}{Z^4}\right) + 15\left(Z^2 + \frac{1}{Z^2}\right) + 20 \text{ (Binomial theorem)} \quad \_\_\_\_\_(1)$$

Also, $Z^m - Z^{-m}$

$= 2i\sin m\theta$

$(2\cos\theta)^6 = (Z-Z^{-1})^6$

$$= Z^6 + \frac{1}{Z^6} - 6\left(Z^4 + \frac{1}{Z^4}\right) + 15\left(Z^2 + \frac{1}{Z^2}\right) - 20 \quad \_\_\_\_\_(2)$$

From (1) and (2),

$2^6(\cos^6\theta - \sin^6\theta) = 2^6(\cos^6\theta + i^6\sin^6\theta)$

$$= 2\left(Z^6 + \frac{1}{Z^6}\right) + 30\left(Z^2 + \frac{1}{Z^2}\right)$$

$= 2(2\cos 6\theta) + 30(2\cos 2\theta)$

$= 4\cos 6\theta + 60\cos 2\theta$

$$\therefore \cos^6\theta - \sin^6\theta = \frac{1}{64}\ [4\cos 6\theta + 60\cos 2\theta]$$

$$= \frac{1}{16}\ (\cos 6\theta + 15\cos 2\theta]$$

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